Viscoelastoplastic Response of Axisymmetric Shells under Impulsive Loadings

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A derivation of finite element equations of and solution to the viscoelastoplastic response of an isotropic axisymmetric shell are presented herein. The generalized Maxwell model is incorporated into the von Mises isotropic yield function. This permits a derivation of the incremental stress as a function of elastic, viscous, and plastic strains. With this relationship inserted into the incremental equation of motion, a direct numerical integration scheme is then used to solve for incremental responses. The plastic tangent stiffness matrix is updated at each incremental time step. Numerical results are presented for a circular plate to verify correctness of the program and subsequently for a spherical cap subjected to uniformly distributed transverse impulsive load of infinite duration.

I. Introduction

APPLICATIONS of shell structures in modern technology have required accurate predictions of the complex structural behavior. The geometric and material nonlinearities coupled with dissipation through time-dependent viscous flow may become significant under either static or dynamic loading.

The mechanical behavior of materials with memory was studied by Green and Rivlin[1] and Green, Rivlin, and Spencer[2] who developed an isothermal theory of viscoelasticity based on the approximation of the stress functional. Such theory, however, is impractical for the material with strong nonlinearities because of the excessive experimental requirements.

Simpler single-integral constitutive equations have been derived from thermodynamic theory by Schapery, with either strains[3] or stresses[4] entering as independent state variables. The uniaxial and multiaxial relations have a form similar to the Boltzmann superposition integral used in linear viscoelasticity.

Special cases of this nonlinear theory were applied with limited success by other investigators to characterize monolithic materials. Leaderman[5] proposed and applied to fibers modified superposition principle. Recently, Schapery[6] and Lou and Schapery[7] studied the type of data needed to evaluate all uniaxial mechanical properties and also the accuracy of the theory.

Perzyna[8] presented the thermodynamic foundations of the theory of viscoplasticity, the essential feature of which is the simultaneous description of rheologic or plastic effects of a material. The simultaneous consideration of viscoelastic and plastic properties of a material is necessitated by the results of experimental investigations of dynamic loads.[9] The viscous properties of the material introduce a history dependence of the states of stress and strain. The plastic properties on the other hand, make these states depend on the path. Different results will be obtained for different deformation paths and for different time duration of the process.

The present study is concerned with the viscoelastoplastic behavior of an isotropic shell under dynamic loads. In view of the fact that no continuum theory on viscoelastoplastic shells which may give closed form solutions is available at present, we must necessarily resort to numerical techniques, preferably the finite element method. Earlier works on finite element applications by Chang[10] and White[11] for linear viscoelasticity, Nickell[12] for one-dimensional viscoelastic wave propagation, Oden and Ramirez[13] for thermomechanical behavior of materials with memory, Malone and Connor[14] for dynamic viscoelasticity and Chung and Eidson[15] for static viscoelastoplastic shell analysis may be noted. In the present analysis, following the plan of Ref. 15, a relaxation kernel expressed in the form of an exponential series[14–16] is used in a numerical time integration process. At each time increment, viscoelastic displacements and stresses are first calculated. These are subsequently used to iterate through incremental plastic loadings based on the Huber-Mises isotropic yield function and associated flow rule. Conditions of yielding are checked through preassigned layers of the shell as well as all elements. The geometric stiffness matrix and the plastic tangent stiffness matrix are updated as iteration cycles progress between integration time increments. A complete response analysis of a circular plate and a spherical cap subjected to uniformly distributed impulsive load of infinite duration is presented as examples.

II. Isotropic Plasticity

The von Mises isotropic yield function F (Ref. 17) may be written

$$F = \bar{\sigma}^2 = 3J \tag{1}$$

where $\bar{\sigma}$ is the equivalent yield stress and J is the second deviatoric stress invariant.

The incremental plastic strain tensor is related by an associated flow rule

$$d\gamma_{\alpha\beta}{}^{(p)} = (\partial F/\partial \sigma^{\alpha\beta})\, d\lambda \tag{2}$$

where the indices α and β range from 1 to 2, and $d\lambda$ is the unknown parameter.

Let the plastic modulus $E_{(p)}$ be given by

$$E_{(p)} = d\bar{\sigma}/d\bar{\gamma}^{(p)} \tag{3}$$

where $d\bar{\gamma}^{(p)}$ is the incremental plastic equivalent yield strain. The incremental plastic work $dW^{(p)}$ is defined

$$dW^{(p)} = \sigma^{\alpha\beta}\, d\gamma_{\alpha\beta}{}^{(p)} = 2\bar{\sigma}^2\, d\lambda = \bar{\sigma}\, d\bar{\gamma}^{(p)}$$

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from which

$$d\lambda = d\bar{\gamma}^{(p)}/2\bar{\sigma} \tag{4}$$

Substituting Eqs. (1) and (4) into Eq. (2) gives

$$d\gamma_{\alpha\beta}{}^{(p)} = Y_{\alpha\beta}\, d\bar{\gamma}^{(p)} \tag{5}$$

In the absence of viscous behavior, the total incremental strain is given by

$$d\gamma_{\alpha\beta} = d\gamma_{\alpha\beta}{}^{(E)} + d\gamma_{\alpha\beta}{}^{(p)} \tag{6}$$

in which (E) and (p) indicate elastic and plastic components, respectively. The incremental elastic strain is, therefore

$$d\gamma_{\alpha\beta}{}^{(E)} = d\gamma_{\alpha\beta} - d\gamma_{\alpha\beta}{}^{(p)} \tag{7}$$

The incremental stress for elastoplastic behavior is given by

$$d\sigma_{(EP)}{}^{\alpha\beta} = E^{\alpha\beta\lambda\mu}(d\gamma_{\alpha\beta} - d\gamma_{\alpha\beta}{}^{(p)}) \tag{8}$$

where $E^{\alpha\beta\lambda\mu}$ is the contravariant tensor of isotropic elasticity moduli. In view of Eqs. (3, 5, and 8), and differentiating Eq. (1), it can easily be shown that

$$d\sigma_{(EP)}{}^{\alpha\beta} = (E^{\alpha\beta\lambda\mu} + \overset{*}{E}{}^{\alpha\beta\lambda\mu})\, d\gamma_{\lambda\mu} \tag{9}$$

where

$$\overset{*}{E}{}^{\alpha\beta\lambda\mu} = E^{\alpha\beta\delta\theta} Y_{\xi\eta}\, Y_{\delta\theta}\, E^{\xi\eta\lambda\mu}/(E_{(p)} + Y_{rs} E^{rsuv} Y_{uv}) \tag{10}$$

which may be interpreted as the contravariant tensor of plastic tangent moduli.

III. Viscoelastoplastic Model

To introduce incremental viscoelastic strains and corresponding stresses in Eq. (9) amounts to adding the time rate of strains into Eq. (9) by defining an appropriate viscoelastic constitutive relationship. To this end, the Boltzmann superposition integral representing the generalized Maxwell model[14,15] is introduced

$$\sigma^{\alpha\beta} = E^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + \eta^{\alpha\beta\lambda\mu}\dot{\gamma}_{\lambda\mu} + \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu} q_{\lambda\mu}{}^{(i)} \tag{11}$$

where stresses and strains are functions of time t, $\eta^{\alpha\beta\lambda\mu}$, and $E_{(i)}{}^{\alpha\beta\lambda\mu}$ are arrays of viscous and spring constants, respectively, the dot implies a time derivative, and the hidden coordinate $q_{\lambda\mu}{}^{(i)}$ is given by

$$q_{\lambda\mu}{}^{(i)} = \int_0^t \exp\left(\frac{-(t-\tau)}{T_{(i)}}\right)\dot{\gamma}_{\lambda\mu}\, d\tau \tag{12}$$

Here, $T_{(i)}$ is the relaxation time and the time rate of strain $\dot{\gamma}_{\lambda\mu}$ may be considered to vary linearly between the discrete time increment Δt (Ref. 15) such that

$$\dot{\gamma}_{\lambda\mu} = \dot{\gamma}_{\lambda\mu}(t-\Delta t) + [\{\tau - (t-\Delta t)\}/\Delta t][\dot{\gamma}_{\lambda\mu}(t) - \dot{\gamma}_{\lambda\mu}(t-\Delta t)] \tag{13}$$

Substituting (13) into (12) gives

$$q_{\lambda\mu}{}^{(i)}(t) = A^{(i)}\dot{X}_{\lambda\mu}{}^{(i)}(t-\Delta t) + B^{(i)}\dot{\gamma}_{\lambda\mu}(t-\Delta t) + C^{(i)}\dot{\gamma}_{\lambda\mu}(t) \tag{14}$$

where

$$A^{(i)} = \exp(-\Delta t/T_{(i)}), \quad B^{(i)} = T_{(i)}[-A^{(i)} + (T_{(i)}/\Delta t)(1 - A^{(i)})]$$

$$C^{(i)} = T_{(i)}[1 - (T_{(i)}/\Delta t)(1 - A^{(i)})]$$

$$\dot{X}_{\lambda\mu}^{(i)}(t-\Delta t) = \sum_{m=1}^{m\Delta t - 1} a_m b_m[\dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\} - m\dot{\gamma}_{\lambda\mu}(m\Delta t) -$$

$$m\dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\} + \sum_{m=1}^{m\Delta t} \frac{a_m c_m}{\Delta t}[\dot{\gamma}_{\lambda\mu}(m\Delta t) - \dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\}]$$

$$a_m = \exp\left(\frac{-m\Delta t}{T_{(i)}}\right), \quad b_m = T_{(i)}\exp\left(\frac{m\Delta t}{T_{(i)}}\right)\left\{1 - \exp\left(\frac{-\Delta t}{T_{(i)}}\right)\right\}$$

$$c_m = T_{(i)}{}^2\exp\left(\frac{m\Delta t}{T_{(i)}}\right)\left[\left(\frac{m\Delta t}{T_{(i)}} - 1\right)\left\{1 - \exp\left(\frac{-\Delta t}{T_{(i)}}\right)\right\} + \frac{\Delta t}{T_{(i)}}\exp\left(\frac{-\Delta t}{T_{(i)}}\right)\right]$$

Here m represents the increment number. The incremental form of (11) for a viscoelastic model is given by

$$d\sigma_{(VE)}{}^{\alpha\beta} = E^{\alpha\beta\lambda\mu}\, d\gamma_{\lambda\mu} + \eta^{\alpha\beta\lambda\mu} d\dot{\gamma}_{\lambda\mu} + \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}(\Gamma_{(t)} + \Gamma_{(t-\Delta t)}\, d\dot{\gamma}_{\lambda\mu})$$

or

$$d\sigma_{(VE)}{}^{\alpha\beta} = E^{\alpha\beta\lambda\mu} d\gamma_{\lambda\mu} + \Lambda^{\alpha\beta\lambda\mu}\, d\dot{\gamma}_{\lambda\mu} \tag{15}$$

where

$$\Lambda^{\alpha\beta\lambda\mu} = \eta^{\alpha\beta\lambda\mu} + \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}(\Gamma_{(t)} + \Gamma_{(t-\Delta t)}) \tag{16}$$

and $\Gamma_{(t)}$ and $\Gamma_{(t-\Delta t)}$ are the terms associated with t and $t - \Delta t$, respectively, in Eq. (14). Adding Eq. (15) and the second term of the right-hand side of Eq. (9) results in the total incremental viscoelastoplastic stress tensor in the form

$$d\sigma^{\alpha\beta} = (E^{\alpha\beta\lambda\mu} + \overset{*}{E}{}^{\alpha\beta\lambda\mu})\, d\gamma_{\lambda\mu} + \Lambda^{\alpha\beta\lambda\mu}\, d\dot{\gamma}_{\lambda\mu} \tag{17}$$

Here $d\gamma_{\lambda\mu}$ contains the elastic and plastic parts of strain and $d\dot{\gamma}_{\lambda\mu}$ is associated with the viscous part of strain.

IV. Finite Element Equations of Motion

The shell theory based on the Love-Kirchhoff hypothesis with small strains, large displacements, and moderate rotations is used in this study. For linear elastic behavior, the stress tensor is given by

$$\sigma^{\alpha\beta} = \sigma_{(m)}{}^{\alpha\beta} + \sigma_{(b)}{}^{\alpha\beta} \tag{18}$$

where the membrane stress tensor $\sigma_{(m)}{}^{\alpha\beta}$ and the bending stress tensor $\sigma_{(b)}{}^{\alpha\beta}$ are, respectively

$$\sigma_{(m)}{}^{\alpha\beta} = hE^{\alpha\beta\lambda\mu}\, e_{\lambda\mu}, \quad \sigma_{(b)}{}^{\alpha\beta} = \frac{h^3}{12}E^{\alpha\beta\lambda\mu}\chi_{\lambda\mu}$$

in which h is the thickness. The membrane strain tensor $e_{\alpha\beta}$ and the bending strain tensor $\chi_{\alpha\beta}$ are of the form, respectively

$$e_{\alpha\beta} = \tfrac{1}{2}\{u_{\alpha|\beta} + u_{\beta|\alpha} - 2u^3 b_{\alpha\beta} + (u_{,\alpha}{}^3 + u^\lambda b_{\alpha\lambda})(u_{,\beta}{}^3 + u^\gamma b_{\beta\gamma})\} \tag{19}$$

$$\chi_{\alpha\beta} = -u_{|\alpha\beta}{}^3 - u_{\lambda|\beta}\, b_\alpha{}^\lambda - u_\lambda\, b_{\alpha|\beta}{}^\lambda - u_\beta{}^\mu u_{\mu|\alpha} + b_\beta{}^\mu b_{\mu\alpha} u^3 \tag{20}$$

Here, all Greek letters range from 1 to 2, u_α and u^3 are the displacements, $b_{\alpha\beta}$ is the second fundamental tensor, the commas and strokes represent, respectively, ordinary and covariant differentiations. It should be also noted that all nonlinear terms in the bending strain are neglected.

At this point, we introduce the local displacement fields over the finite element

$$\theta_i = \psi_{iN}\, \theta^N \tag{21}$$

where θ_i represents the ith generalized displacement coordinate within the element, ψ_{iN} is the normalized local interpolation function,[18] and θ^N is the component of generalized coordinates at all nodes of the element. In our example problem, we discuss an axisymmetric shell with one-dimensional meridional line element with 4 degrees-of-freedom at each node to include linear variations of meridional and tangential displacements and a cubic variation of transverse displacement, and the meridional rotation.

The general finite element equation of motion is given by[18,19]

$$m_{NM}\, \ddot{\theta}^M + \int_A \sigma_{(m)}{}^{\alpha\beta}\frac{\partial e_{\alpha\beta}}{\partial\theta^N}\, dA + \int_A \sigma_{(b)}{}^{\alpha\beta}\frac{\partial\chi_{\alpha\beta}}{\partial\theta^N}\, dA = P_N \tag{22}$$

Here, m_{NM} is the mass matrix,

$$m_{NM} = h\int_A \rho\psi_N{}^i\psi_{iM}\, dA \tag{23}$$

where ρ is the density and dA is the differential surface area, and p_N are the components of generalized force at a node.

The total strain tensor is given by

$$\gamma_{\alpha\beta} = e_{\alpha\beta} + z\chi_{\alpha\beta} \tag{24}$$

in which z is the thickness coordinate. In view of Eqs. (19–21, and 24), we obtain

$$e_{\alpha\beta} = A_{N\alpha\beta}\, \theta^N + C_{NM\alpha\beta}\, \theta^M\theta^N, \quad \chi_{\alpha\beta} = B_{N\alpha\beta}\, \theta^N$$

and

$$\partial e_{\alpha\beta}/\partial\theta^N = A_{N\alpha\beta} + C_{NM\alpha\beta}\, \theta^M, \quad \partial\chi_{\alpha\beta}/\partial\theta^N = B_{N\alpha\beta}$$

where $A_{N\alpha\beta}$, $C_{NM\alpha\beta}$, $B_{N\alpha\beta}$ represent derivatives of the normalized interpolation function in the membrane and bending strains of Eqs. (19) and (20).

Introducing a perturbation or an increment to displacements and forces such that $d\theta^N = \Theta^N$, $dp_N = P_N$ and

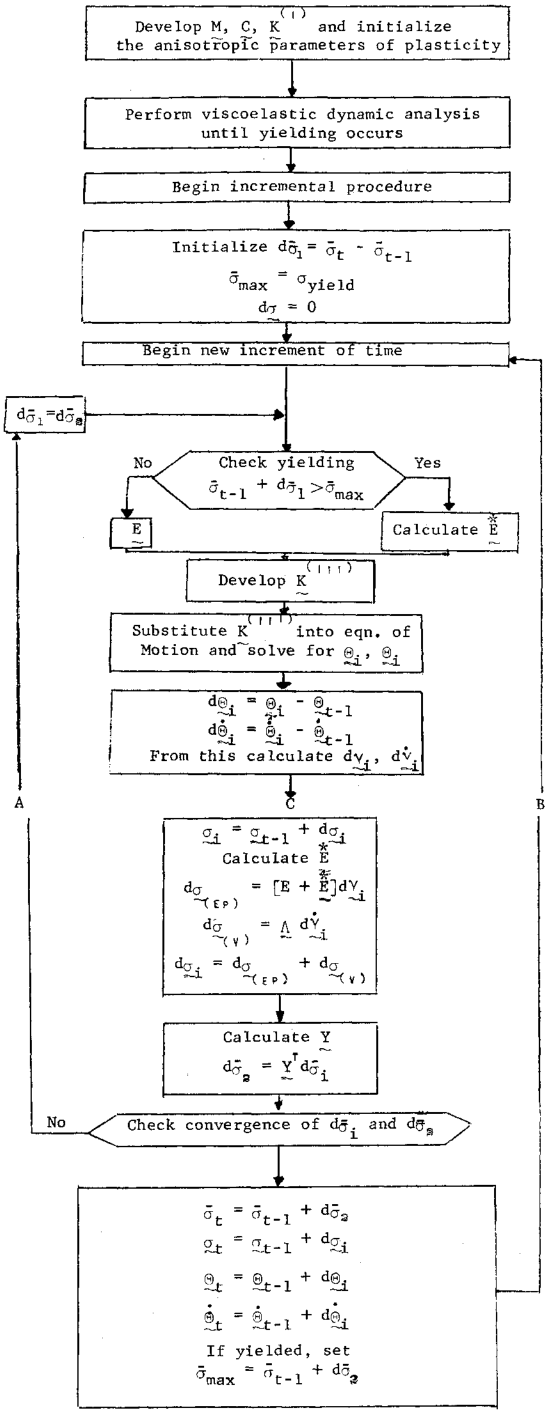

Fig. 1 Procedure for dynamic analysis of viscoelastoplastic anisotropic shells.

$$de_{\alpha\beta} = (A_{N\alpha\beta} + C_{NM\alpha\beta}\,\theta^M)\Theta^N, \quad d\chi_{\alpha\beta} = B_{N\alpha\beta}\,\Theta^N,$$
$$d\left(\frac{\partial e_{\alpha\beta}}{\partial \theta^N}\right) = C_{NM\alpha\beta}\,\Theta^M, \quad d\left(\frac{\partial \chi_{\alpha\beta}}{\partial \theta^N}\right) = 0$$

we can now write Eq. (22) in the incremental form,

$$m_{NM}\,\ddot{\Theta}^M + \int_A \sigma_{(m)}{}^{\alpha\beta} C_{NM\alpha\beta}\, dA\,\Theta^M + \int_A d\sigma_{(m)}{}^{\alpha\beta}(A_{N\alpha\beta} + C_{NM\alpha\beta}\,\theta^M)\,dA + \int_A d\sigma_{(b)}{}^{\alpha\beta} B_{N\alpha\beta}\, dA = P_N \quad (25)$$

in which $d\sigma_{(m)}{}^{\alpha\beta}$ and $d\sigma_{(b)}{}^{\alpha\beta}$ are deduced from Eq. (17) as

$$d\sigma_{(m)}{}^{\alpha\beta} = h(E^{\alpha\beta\lambda\mu} + \overset{*}{E}{}^{\alpha\beta\lambda\mu})\,de_{\lambda\mu} + h\Lambda^{\alpha\beta\lambda\mu}\,d\dot{e}_{\lambda\mu} \quad (26)$$

and

$$d\sigma_{(b)}{}^{\alpha\beta} = \frac{h^3}{12}(E^{\alpha\beta\lambda\mu} + \overset{*}{E}{}^{\alpha\beta\lambda\mu})\,d\chi_{\lambda\mu} + \left(\frac{h^3}{12}\right)\Lambda^{\alpha\beta\lambda\mu}\,d\dot{\chi}_{\lambda\mu} \quad (27)$$

With Eqs. (26) and (27) substituted in Eq. (25), we arrive at the final incremental matrix form of equation of motion

$$\mathbf{m}\ddot{\Theta} + \mathbf{C}\dot{\Theta} + \mathbf{K}^{(I)}\Theta = \mathbf{F} \quad (28)$$

where $\mathbf{m}$ is given by Eq. (23). The viscosity matrix $\mathbf{C}$ is given by, neglecting higher order terms

$$C_{NM} = h\int_A \left(\eta^{\alpha\beta\lambda\mu} + \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}\Gamma_{(t)}\right) A_{N\alpha\beta}\, A_{M\lambda\mu}\, dA + \frac{h^3}{12}\int_A \left(\eta^{\alpha\beta\lambda\mu} + \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}\Gamma_{(t)}\right) B_{N\alpha\beta}\, B_{M\lambda\mu}\, dA \quad (29)$$

The linear elastic stiffness matrix $\mathbf{K}^{(I)}$ is, similarly

$$K_{NM}{}^{(I)} = h\int_A E^{\alpha\beta\lambda\mu} A_{N\alpha\beta}\, A_{M\lambda\mu}\, dA + \frac{h^3}{12}\int_A E^{\alpha\beta\lambda\mu} B_{N\alpha\beta}\, B_{M\lambda\mu}\, dA$$

The incremental load vector $\mathbf{F}$ is of the form

$$\mathbf{F} = \mathbf{P} + \mathbf{L}$$

where $\mathbf{P} = \mathbf{P}_A + \mathbf{K}^{(II)}\Theta_{(t-1)} + \mathbf{K}^{(III)}\Theta_{(t-1)}$ and $\mathbf{P}_A$ = Applied dynamic load within a time increment. The geometric stiffness matrix $\mathbf{K}^{(II)}$ and the plastic tangent stiffness matrix $\mathbf{K}^{(III)}$ are, respectively

$$K_{NM}{}^{(II)} = \int_A \sigma_{(m)}{}^{\alpha\beta} C_{NM\alpha\beta}\, dA$$

$$K_{NM}{}^{(III)} = h\int_A \overset{*}{E}{}^{\alpha\beta\lambda\mu} A_{N\alpha\beta}\, A_{M\lambda\mu}\, dA + \frac{h^3}{12}\int_A \overset{*}{E}{}^{\alpha\beta\lambda\mu} B_{N\alpha\beta}\, B_{M\lambda\mu}\, dA$$

The higher order terms are neglected in the last expression. Note also that the subscript $t-1$ indicates the previous time increment, suggesting that both the geometric stiffness matrix and the plastic tangent stiffness matrix contribute to equivalent loadings with one time increment lagging out of phase. Finally, the viscosity equivalent load $\mathbf{L}$ is given by, neglecting higher order terms

$$L_N = \left[h\int_A \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}\Gamma_{(t-1)} A_{N\alpha\beta}\, A_{M\lambda\mu}\, dA + \frac{h^3}{12}\int_A \sum_{i=1}^{n} E_{(i)}{}^{\alpha\beta\lambda\mu}\Gamma_{(t-1)} B_{N\alpha\beta}\, B_{M\lambda\mu}\, dA\right]\dot{\Theta}^M_{(t-1)} \quad (30)$$

The number of Maxwell elements in Eqs. (29) and (30) depend on the experimental evidence.

V. Direct Integration of Equations of Motion and Iterative Solution

Various direct numerical integration schemes are available in the literature.[20] In this study the constant acceleration method is used. The recurrence formula may be written in the form

$$[\mathbf{m} + (\Delta t/2)\mathbf{C} + (\Delta t^2/4)\mathbf{K}^{(I)}]\ddot{\Theta}_{(t)} = \mathbf{F} - \mathbf{R} \quad (31)$$

where Δt is the time increment within which $\ddot{\Theta}$ is considered constant, and

$$\mathbf{R} = \mathbf{C}[\dot{\Theta}_{(t-1)} + (\Delta t/2)\ddot{\Theta}_{(t-1)}] + \mathbf{K}^{(I)}[\Theta_{(t-1)} + \Delta t\dot{\Theta}_{(t-1)} + (\Delta t^2/4)\ddot{\Theta}_{(t-1)}] \quad (32)$$

Initially, all terms associated with $(t-1)$ in $\mathbf{F}$ and $\mathbf{R}$ are equal to zero. Therefore, it is possible to solve for $\ddot{\Theta}_{(t)}$ from Eq. (31). The incremental velocities and displacements are calculated from

$$\dot{\Theta}_{(t)} = \dot{\Theta}_{(t-1)} + (\Delta t/2)\ddot{\Theta}_{(t-1)} + (\Delta t/2)\ddot{\Theta}_{(t)} \quad (33)$$

$$\Theta_{(t)} = \Theta_{(t-1)} + \Delta t\dot{\Theta}_{(t-1)} + (\Delta t^2/4)\ddot{\Theta}_{(t-1)} + (\Delta t^2/4)\ddot{\Theta}_{(t)} \quad (34)$$

With these results $\mathbf{F}$ and $\mathbf{R}$ can be calculated and substituted in Eqs. (31, 33, and 34) repeatedly until converged response and stresses are obtained within the time increment. This process may continue to any desired number of time increments. Detailed descriptions of this procedure are given below (see also Fig. 1 for flow chart).

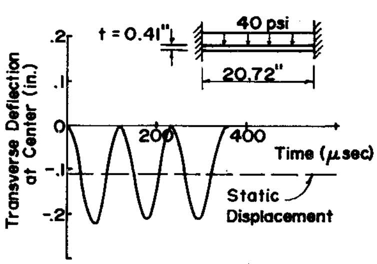

Fig. 2 Viscoelastoplastic transverse displacement at center of fixed-edge isotropic circular plate under uniformly distributed transverse dynamic load of infinite duration.

a) During the first time increment, the generalized displacements and stresses depend on the mass matrix, viscosity matrix, linear elastic stiffness matrix, and applied dynamic load only.

b) Examine the current viscoelastic stresses in (a) and determine first of all if any portion through thickness of any element has yielded. If so, the plastic tangent stiffness coefficient for that portion is developed.

c) The plastic tangent stiffness matrix, if nonzero, is incorporated into the recurrence formula of Eq. (31) and then generalized displacements and velocities are the difference between the values for the current and previous cycles.

d) From the incremental displacements and velocities the incremental strains and strain rates are determined.

e) The incremental equivalent stress is calculated and compared with the incremental equivalent stress of the previous cycle.

f) Steps (c–e) are repeated until convergence, or a certain desired accuracy is obtained for the incremental equivalent yield stress.

g) For a yielded element (or layer of the element), the maximum equivalent stress which was originally set equal to the input yield stress is now updated by adding the incremental equivalent yield stress to account for strain hardening. The incremental viscous stresses are incorporated during the first cycle through iteration but excluded from further cycles for plastic yielding.

h) If no yielding occurred anywhere in the shell the steps (b–g) are omitted.

i) A new time increment is initiated and the above steps are repeated until desired time increments have been completed.

VI. Applications

The theory and procedures as described earlier have been applied to a number of example problems. The isotropic linear elastic and geometric nonlinear cases were compared with the results of others[21,22] and a close agreement was verified. Here, typical viscoelastoplastic responses for a circular plate and a spherical cap are presented.

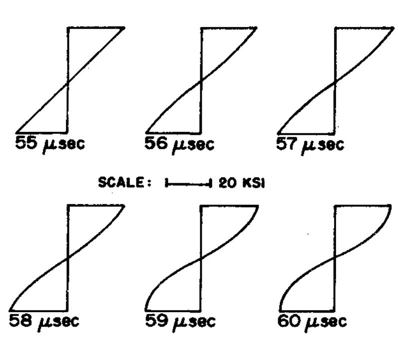

Fig. 3 Radial stress variation through thickness at center of plate in Fig. 2.

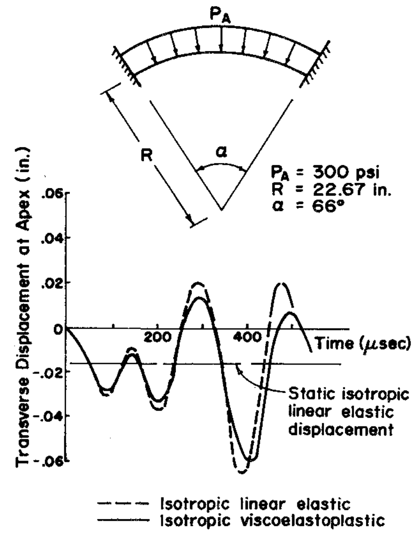

Fig. 4 Transverse displacement at apex of an isotropic viscoelastoplastic spherical cap subjected to uniformly distributed transverse load of infinite duration.

The viscoelastoplastic transverse displacements at the center of a fixed-edge isotropic circular plate subjected to the uniformly distributed transverse impulsive load of 40 psi of infinite duration are shown in Fig. 2. Material properties used are: modulus of elasticity = 0.105×10^8 psi; Poisson's ratio = 0.3; plastic modulus $E_{(p)}$ = 210,000 psi; equivalent yield stress = 22,000 psi; viscosity constant $\eta_{(i)}$ = 6.56 psi-sec; spring constant $E_{(i)} = 0.105 \times 10^8$ psi. The incremental time step, Δt is taken at 10 μsec. The radial stress variation at the center through the thickness is shown in Fig. 3. Gradual changes of nonlinearity are well demonstrated between the time steps of 55 μsec and 60 μsec. Unfortunately, no comparison can be made at the present as no such data by other investigators are available to the authors.

As a second example a spherical cap as shown in Fig. 4 subjected to externally applied transverse impulsive load of 300 psi of infinite duration was analyzed. Six layers through thickness and 5 elements were used. The material data are

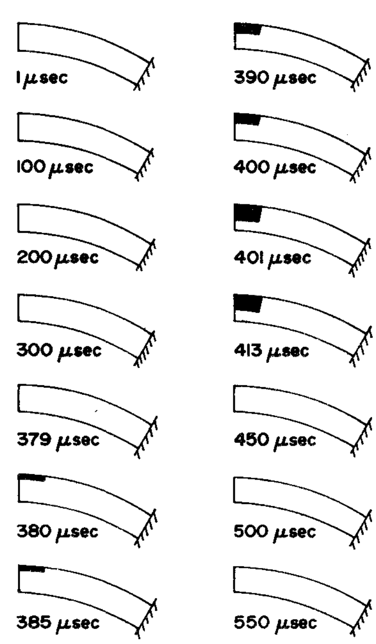

Fig. 5 Yielded region for the spherical cap in Fig. 4.

identical to those for the circular plate except that the equivalent yield stress of 24,000 psi is used. The idea is to choose an adequate equivalent yield stress for a given applied load intensity such that neither excessive nor little yielding occurs throughout the time steps under study. It has been observed that if too many layers or too many elements yield at the same time, iteration cycles required for convergence increase considerably. In the case like this convergence may prevail promptly if the time increment is decreased. To this end, $\Delta t = 1$ μsec was used. In Fig. 4, the transverse displacements at apex for the cases of isotropic linear elasticity and isotropic viscoelastoplasticity are compared. It may be argued that larger displacements occur due to plasticity, but the contrary is true for viscosity. Even for such a small value of viscosity constant at 6.56 psi-sec, the damping effect under dynamic load is so predominant that the viscoelastoplastic response is considerably lower than the linear elastic response. Finally, the yielded regions are shown by shaded area in Fig. 5. Yielding occurs for the first time on the top layer of the element close to the apex at 380 μsec after application of the load, and gradually spreads to underlying layers. However, these yielded layers are completely unloaded at 413 μsec. It is interesting to note that such yielding occurred shortly before and after the peak response at 405 μsec. Behavior beyond 550 μsec was not studied because of the computer time limitation.

VII. Concluding Remarks

A powerful and efficient method of dynamic analysis of isotropic viscoelastoplastic shell has been introduced. An evaluation of accuracy cannot be made at present as no previous work either analytical or experimental is available to the authors. Based on the present study the following conclusions are drawn: 1) effect of viscosity is significant under dynamic load giving considerably lower response; 2) plastic yielding occurs sometime after application of the load, corresponding to peak responses; 3) such yielding is soon recovered by unloading.

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